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INTRODUCTION TO UNCERTAINTY ANALYSIS

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WHAT IS MEASUREMENT UNCERTAINTY?

- Measurement result: **number** × **unit**
  - e.g. $L = 2.00$ m
- Measurement uncertainty: **number** × **unit, probability**
  - e.g. $U(L) = 0.02$ m at 95 % confidence
- Relative uncertainty: **number, probability**
  - e.g. $U_r(L) = U(L)/L = 0.01$ at 95 % confidence
- “Error” is not the same as uncertainty
- “Accuracy” is not a quantitative term

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TYPES OF UNCERTAINTY

- Random
  - repeated measurements give randomly different results
  - uncertainty reduced by averaging many readings
- Systematic
  - all results biased in the same way
  - uncertainty not reduced by averaging readings

- Type A
  - Estimated by statistical analysis
- Type B
  - Estimated by other means

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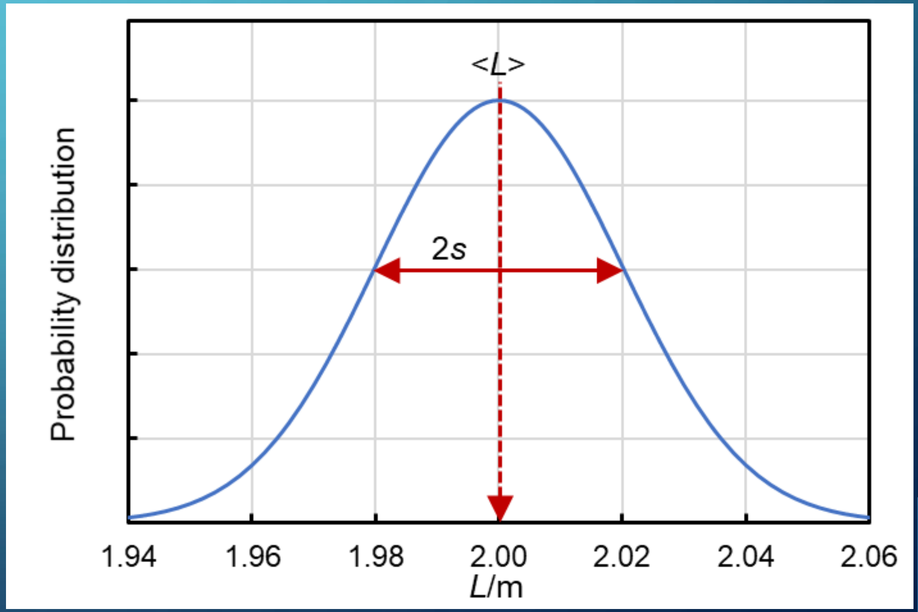
STANDARD UNCERTAINTY FOR TYPE A EVALUATION

- Assuming a normal probability distribution
- Make n observations of a quantity x
- Calculate the mean $\langle x \rangle$ and standard deviation s
- Standard uncertainty $u(x) = s/\sqrt{n}$
- Reduced by increasing n



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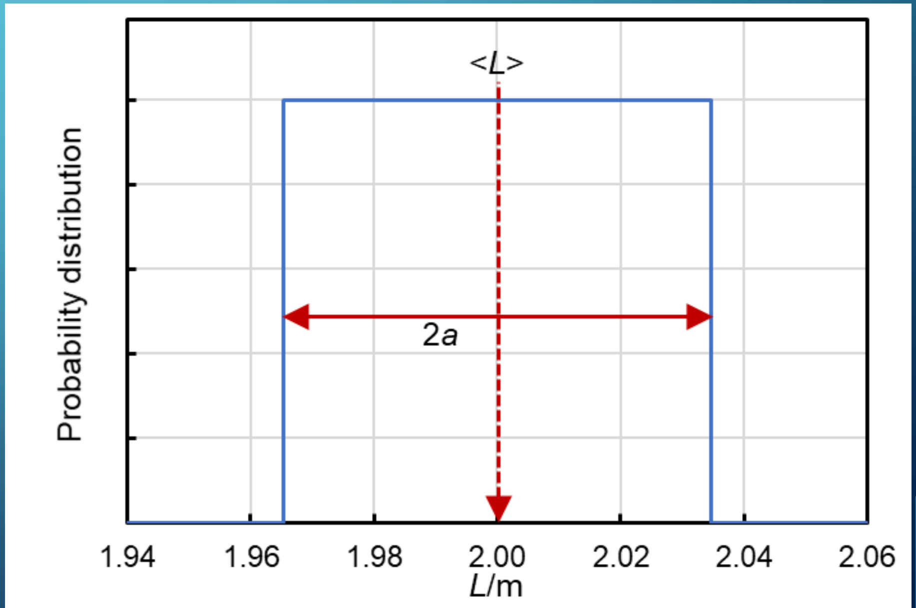
STANDARD UNCERTAINTY FOR TYPE B EVALUATION

- Assuming a rectangular probability distribution
- Establish centre value $<x>$ and half width a of the distribution
- Standard uncertainty $u(x) = a/\sqrt{3}$



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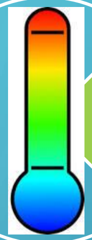
INFLUENCE FACTORS

Length



Temperature

- Often the quantity to be measured is affected by influence factors
  - e.g. the length L of a rod is influenced by temperature T
- Uncertainties in the values of these influence factors affect the overall uncertainty of the measurement
- Must convert all uncertainties to the same units
  - e.g. contribution of standard uncertainty in temperature to combined standard uncertainty of length L is $(\partial L/\partial T)\cdot u(T) = \alpha \cdot L \cdot u(T)$

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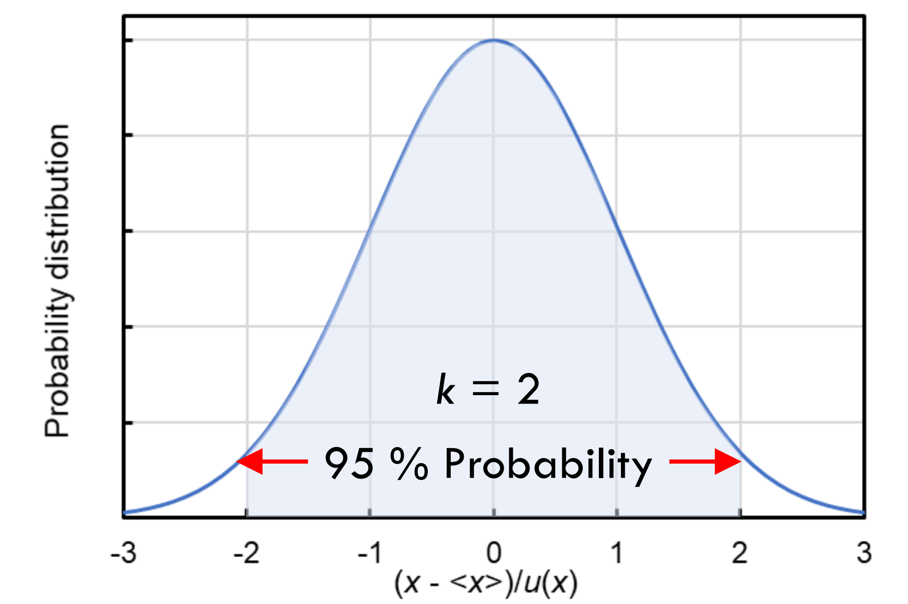
COMBINING UNCERTAINTIES

- Uncorrelated, random uncertainties combine in quadrature
- For example, length measurement with type A standard uncertainties:
  - $u(L)$ in direct measurement
  - $u(T)$ in temperature
- Overall combines standard uncertainty $u_c(L)$ given by

  $[u_c(L)]^2 = [u\,(L)]^2 + [(\partial L/\partial T)\cdot u(T)]^2$
- Usually include Type B uncertainties in the same way through additional terms

CONFIDENCE INTERVAL

- Establish the combined overall standard uncertainty $u_c(x)$
- Select a coverage factor k, e.g.:
  - $k = 1$ for 67 % confidence
  - $k = 2$ for 95 % confidence
  - $k = 3$ for 99.7 % confidence
- Overall uncertainty $U(x) = k \cdot u(x)$
- Confidence interval $\langle x \rangle \pm k \cdot u(x)$
- e.g. for a length: "$L = (2.00 \pm 0.02)$ m at 95 % confidence"



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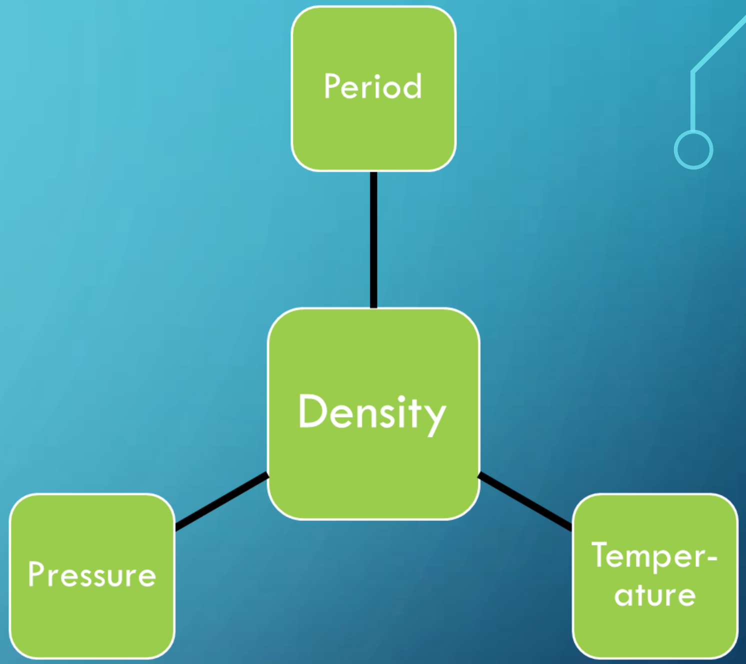
EXAMPLE UNCERTAINTY BUDGET: LENGTH OF A ROD*

Source of uncertainty	Value	Distribution	Divisor	Sensitivity factor	Standard uncertainty
Calibration uncertainty (tape measure)	2.0 mm	Normal	2	1	1.0 mm
Resolution	0.5 mm	Rectangular	$\sqrt{3}$	1	0.3 mm
Repeatability	0.7 mm	Normal	1	1	0.7 mm
Temperature	2 K	Rectangular	$\sqrt{3}$	0.17 mm K^{-1}	0.2 mm
Combined standard uncertainty		Assumed normal			1.3 mm
Expanded uncertainty		Assumed normal $k = 2$			2.6 mm

* Nylon rod, nominal length 2 m long

MORE COMPLEX MEASUREMENTS

- A tape measure returns a more-or-less direct measurement of length
- Other measurement systems rely on a physical or empirical model to relate measurement of one or more quantities to the quantity of interest
- e.g. In a vibrating-tube densimeter, one measures period of oscillation, which is related to density by a calibration equation
- The analysis is then similar but more complex



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SUGGESTED READING

- Stephanie Bell "The Beginner's Guide to Uncertainty of Measurement" (NPL, 1999) http://publications.npl.co.uk/npl_web/pdf/mgpg11.pdf
- Barry N. Taylor and Chris E. Kuyatt "Guidelines for Evaluating and Expressing the Uncertainty of NIST Measurement Results" NIST Technical Note 1297 (NIST, 1994) https://www.nist.gov/pml/nist-technical-note-1297
- "Evaluation of measurement data — Guide to the expression of uncertainty in measurement" (GUM 1995) (JCGM, 2010) https://www.bipm.org/utils/common/documents/jcgm/JCGM_100_2008_E.pdf

PART II

FURTHER DETAILS

COMBINING STANDARD UNCERTAINTIES

- Generally, random uncertainties combine according to the relation

$$u_c^2(f) = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)$$
$$= \sum_{i=1}^{N}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)$$

where $f(x_1, x_2, x_3, \cdots x_N)$ is the quantity being determined, x_i ($i = 1, 2, 3 \cdots N$) are the independent or input variables being measured, and $u(x_i, x_j)$ is the covariance between x_i and x_j.

- $u(x_i, x_j)$ is known as the variance-covariance matrix. If $i = j$ then $u(x_i, x_j) = u^2(x_i)$
- Covariance measures coupling between random variation of two variables

COMBINING STANDARD UNCERTAINTIES

- If the independent variables are uncorrelated then the covariances between x_i and x_j vanish for $i \neq j$ and the expression reduces to

$$u_c^2(f) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i)$$

where $u\,(x_i)$ standard uncertainty of input variable and $u^2(x_i)$ is the variance

- This is the basis of the relation given on slide 7
- It is often assumed that only these diagonal elements of the variance-covariance matrix terms are significant; however, that is not always true.

OVERALL RELATIVE STANDARD UNCERTAINTIES

- Assuming uncorrelated independent variables we have

$$u_c^2(f) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i)$$

- Often convenient to work with relative uncertainties $u_r(x_i) = u(x_i)/x_i$
- In that case,

$$u_{r,c}^2(f) = \sum_{i=1}^{N} \left(\frac{1}{f} \frac{\partial f}{\partial x_i} \right)^2 \left[x_i u_r(x_i) \right]^2$$
$$= \sum_{i=1}^{N} \left(\frac{\partial \ln f}{\partial \ln x_i} \right)^2 u_r^2(x_i)$$

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EXAMPLE CALCULATION: VIBRATING-TUBE DENSIMETER

- Working equation: $\rho = A\tau^2 - B$

  τ = period of oscillation.

  A and B calibration parameters

- Expanded working equation: $$\rho = \frac{\rho_1\left(\tau^2 - \tau_2^2\right) - \rho_2\left(\tau^2 - \tau_1^2\right)}{\left(\tau_1^2 - \tau_2^2\right)}$$

- Input variables to determine $\rho(T,p)$ are: ρ_1, ρ_2, τ_1, τ_2, τ, T, p; therefore:

$$u^2(\rho) = \left[\left(\frac{\partial\rho}{\partial\tau}\right)u(\tau)\right]^2 + \left[\left(\frac{\partial\rho}{\partial\tau_1}\right)u(\tau_1)\right]^2 + \left[\left(\frac{\partial\rho}{\partial\tau_2}\right)u(\tau_2)\right]^2 + \left[\left(\frac{\partial\rho}{\partial\rho_1}\right)u(\rho_1)\right]^2 + \left[\left(\frac{\partial\rho}{\partial\rho_2}\right)u(\rho_2)\right]^2 + \left[\left(\frac{\partial\rho}{\partial T}\right)u(T)\right]^2 + \left[\left(\frac{\partial\rho}{\partial p}\right)u(p)\right]^2 + u_B^2(\rho)$$

EXAMPLE CALCULATION: VIBRATING-TUBE DENSIMETER

$$u^2(\rho)=\left[\left(\frac{\partial\rho}{\partial\tau}\right)u(\tau)\right]^2+\left[\left(\frac{\partial\rho}{\partial\tau_1}\right)u(\tau_1)\right]^2+\left[\left(\frac{\partial\rho}{\partial\tau_2}\right)u(\tau_2)\right]^2+\left[\left(\frac{\partial\rho}{\partial\rho_1}\right)u(\rho_1)\right]^2+\left[\left(\frac{\partial\rho}{\partial\rho_2}\right)u(\rho_2)\right]^2+\left[\left(\frac{\partial\rho}{\partial T}\right)u(T)\right]^2+\left[\left(\frac{\partial\rho}{\partial p}\right)u(p)\right]^2+u_B^2(\rho)$$

- $u(\tau)$, $u(\tau_1)$, $u(\tau_2)$ uncertainties of the period during measurement in th test fluid, calibration fluid 1 and calibration fluid 2
- $u(\rho_1)$, $u(\rho_2)$ uncertainties in the densities of calibration fluids 1 and 2, including the propagated effects of temperature and pressure uncertainties during calibration
- $u(T)$, $u(p)$ standard uncertainties in temperature and pressure during measurement
- $u_B(\rho)$ type B standard uncertainty

EXPLICIT EXPRESSION

First line from differentiation of the working equation

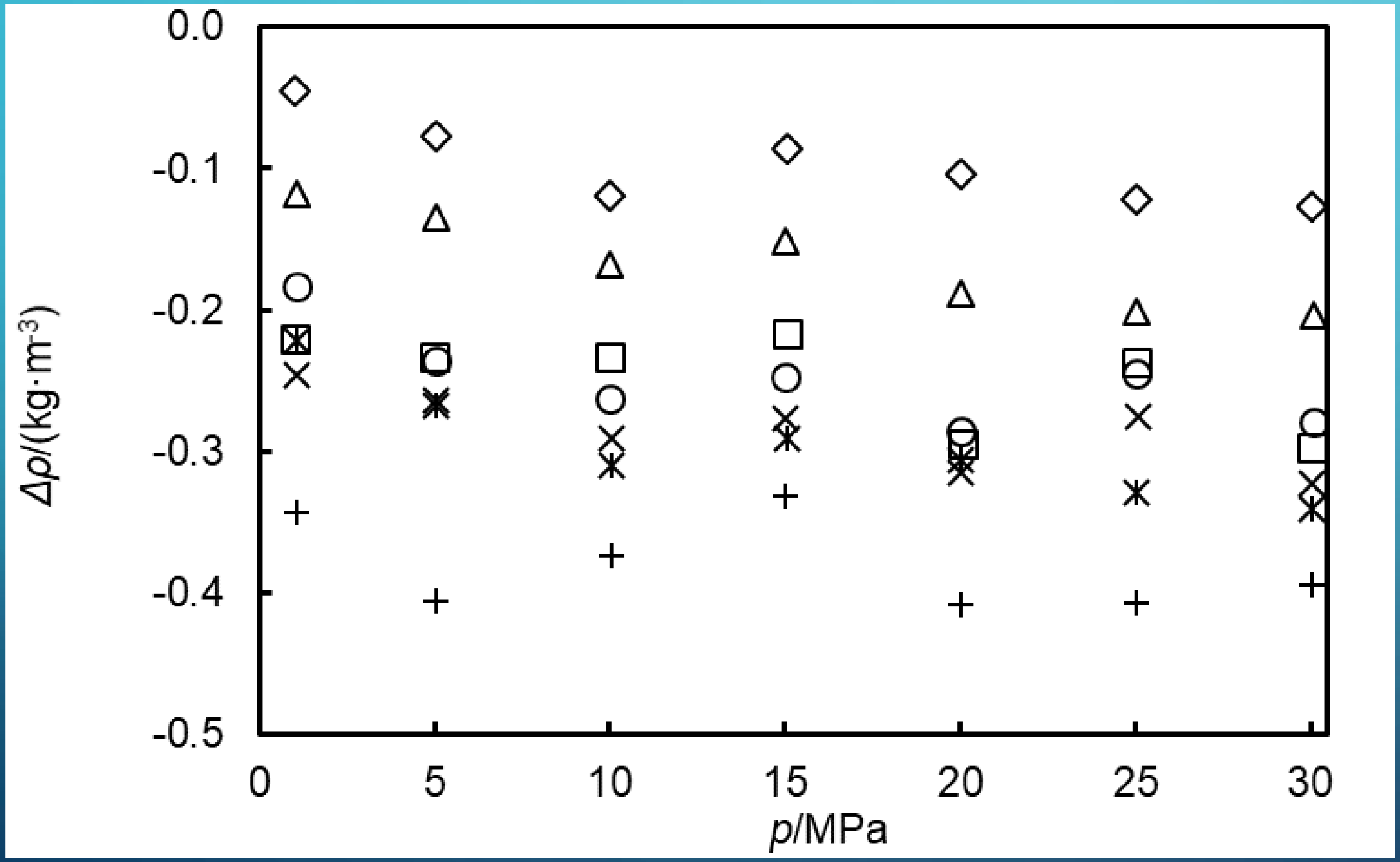
$$u_c^2(\rho) = \left[\frac{2\tau(\rho_1-\rho_2)}{\tau_1^2-\tau_2^2}u(\tau)\right]^2 + \left[\frac{2\tau_1(\rho_2-\rho)}{\tau_1^2-\tau_2^2}u(\tau_1)\right]^2 + \left[\frac{2\tau_2(\rho-\rho_1)}{\tau_1^2-\tau_2^2}u(\tau_2)\right]^2 + \left[\frac{\tau^2-\tau_2^2}{\tau_1^2-\tau_2^2}u(\rho_1)\right]^2 + \left[\frac{\tau_1^2-\tau^2}{\tau_1^2-\tau_2^2}u(\rho_2)\right]^2$$

$$+\left[\left(\frac{\partial\rho}{\partial p}\right)_T u(p)\right]^2 + \left[\left(\frac{\partial\rho}{\partial T}\right)_p u(T)\right]^2 + u_B^2(\rho)$$

These terms to be estimated from either measured data at different T, p conditions or from an independent model (e.g. an equation of state)

Type-B term also to be estimated (see next slide)

ESTIMATING THE TYPE B UNCERTAINTY



- Comparison with independent measurements of lower uncertainty
- Tested here on methylbenzene
- Comparison with NIST measurements with < 0.02 % relative uncertainty
- Estimate $u_B = 0.26\ \mathrm{kg\cdot m^{-3}}$
- Note that the vibrating-tube data are systematically lower than the reference data

SUMMARY OF UNCERTAINTY CALCULATION: NONANE

Quantity	Value	Standard uncertainty	Sensitivity factor	Standard uncertainty
ρ_1	34.92 kg·m^{-3}	0.17 kg·m^{-3}	0.31	0.011 kg·m^{-3}
ρ_2	971.82 kg·m^{-3}	0.10 kg·m^{-3}	0.70	0.019 kg·m^{-3}
τ_1	2479.08 μs	0.01 μs	1.41 kg·m^{-3}·μs^{-1}	0.014 kg·m^{-3}
τ_2	2673.25 μs	0.01 μs	3.48 kg·m^{-3}·μs^{-1}	0.035 kg·m^{-3}
τ	2615.64 μs	0.01 μs	4.90 kg·m^{-3}·μs^{-1}	0.049 kg·m^{-3}
p	30 MPa	0.035 MPa	0.90 kg·m^{-3}·MPa^{-1}	0.031 kg·m^{-3}
T	373.15 K	0.025 K	0.65 kg·m^{-3}·K^{-1}	0.016 kg·m^{-3}
Type B				0.26 kg·m^{-3}
Expanded uncertainty (k = 2)			**0.54 kg·m^{-3}**	

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WHAT IS MY INPUT VARIABLES ARE CORRELATED

- It is possible to apply the generic formula directly (equation on slide 13)
- Often difficult to evaluate the correlation between inputs analytically
- Alternative: Monte Carlo simulation
- Require a mathematical model for $f = f(x_1, x_2, x_3, \cdots x_N)$
- Randomly select an input variables x_i, perturb it by $u(x_i)$ and find the change δf in the function allowing for any dependence of the other variables on x_i
- Repeat a very large number of times and determine $u^2(f) = \langle(\delta f)^2\rangle$